# PHYS 798C Spring 2022
# Lecture 16 Summary

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## I. SUPERCONDUCTORS IN A MAGNETIC FIELD - DOMAIN WALL ENERGY

Here we begin to consider how a superconductor compromises with a magnetic field.

### A. Intermediate State

Superconductors of type-I will go in to an "intermediate state" when exposed to a magnetic field. The magnetic field is channeled in to normal domains, where the magnetic field value is the thermodynamic critical field $H_c$, leaving other regions in the Meissner state with $H = 0$. This lamenar structure can be quite complicated, depending on sample geometry and pinning sites (to be discussed later). Pictures of these intricate domain structures are available on the class web site.
Superconductors of type-II divide the magnetic field into finer and finer pieces until each carries one flux quantum $\Phi_0 = h/2e$ in a region surrounded by superconducting material.

The difference between these two cases comes about from the energy cost of creating a superconductor / normal (S/N) interface. Consider the order parameter and magnetic field profile in a cross section through the S/N interface, and call this coordinate the $x$ direction. In the type-I case (defined by $\kappa = \lambda_{eff}/\xi_{GL} << 1$) the magnetic field persists well into the superconducting side of the interface, despite the fact that the GL order parameter is suppressed over a long length scale there. This situation, namely the loss of condensation energy over an extended length and at the same time the generation of strong screening currents to shield out the magnetic field on short length scales, gives rise to a net positive energy cost for creation of the S/N interface.

In the type-II case (defined by $\kappa = \lambda_{eff}/\xi_{GL} >> 1$) the GL order parameter persists right up to the interface, whereas the magnetic field is suppressed over a long length scale. This yields a net negative interface energy, meaning that such interfaces will proliferate spontaneously. Their proliferation is stopped only by the quantum limit of magnetic field dilution, when each "normal region" carries exactly one quantum of magnetic flux.

One can define the domain wall energy per unit area as follows,
$\gamma = \frac{\mu_0 H_c^2}{2}\delta$, where $\delta$ is the domain wall thickness, defined as
$\delta = \int_{-\infty}^{\infty}\left[\left(1-\frac{h(x)}{H_c}\right)^2 - \left(\frac{\psi(x)}{\psi_\infty}\right)^4\right]dx$, where $h(x)$ is the microscopic magnetic field. Note that $\delta$ can be positive or negative, depending on the competition between magnetic and condensation energies.
The integrand is zero both in the uniform normal metal and the uniform superconductor, picking up contributions only in the S/N interface.

Take the origin ($x = 0$) to be at the normal metal side of the interface. Using the following field and order parameter profiles we can calculate the S/N energy:

$$h(x) = \begin{cases} H_c e^{-x/\lambda} & (x > 0) \\ H_c & (x < 0) \end{cases}$$

$$\psi(x) = \begin{cases} \psi_\infty(1 - e^{\sqrt{2}x/\xi}) & (x > 0) \\ 0 & (x < 0) \end{cases}$$

Note that this $h(x)$ and $\psi(x)$ profile was not determined self-consistently from the two coupled nonlinear GL equations. Hence it will not give exact results for the $\delta$ calculation.
The outcome for the S/N interace thickness is,
Type-I $\kappa << 1$ has $\delta = \frac{25}{24}\sqrt{2}\xi \approx 1.47\xi$. The exact GL result is $\frac{4\sqrt{2}}{3}\xi \approx 1.89\xi$.
Type-II $\kappa >> 1$ has $\delta = -\frac{3}{2}\lambda$. The exact GL result is $-1.104\lambda$.

The crossover in this approximate calculation $\delta = 0$ occurs at $\kappa = \frac{25\sqrt{2}}{36} = 0.98$
The exact value for the crossover between type-I and type-II is at $\kappa = 1/\sqrt{2}$.

### B. Critical current of a filamentary superconductor

Consider a very thin quasi-one-dimensional cylindrical current-carrying superconducting filament. The diameter $d$ is much less than the GL coherence length $\xi_{GL}$. In this case it is unlikely that the order parameter will vary across the width of the wire, and also unlikely it will vary longitudinally if the wire is sufficiently short. In this case we can take the magnitude of the order parameter as constant, and attribute all of the spatial variation to the phase,
$\psi(x) = |\psi| e^{i\phi(r)}$, where $|\psi|$ is constant. In this case the current is given by,
$\vec{J} = \frac{e^*}{m^*}|\psi|^2 \left(\hbar\vec{\nabla}\phi - e^*\vec{A}\right)$,
$= e^*|\psi|^2\vec{v}_s$

Minimizing the free energy expansion with respect to $|\psi|$ yields,
$|\psi|^2 = |\psi_\infty|^2 \left[1 - \left(\frac{m^*\xi_{GL}(T)v_s}{\hbar}\right)^2\right]$
The GL order parameter is suppressed in the presence of a super-current. This is often refered to as current-induced "de-pairing", although there is no notion of pairing in GL theory. Note that the GL coherence length diverges as $T_c$ is approached. This means that near $T_c$ even a small current will be sufficient to produce significant de-pairing, making the superconductor very nonlinear.

The resulting super-current is now nonlinear in the superfluid velocity:
$J_s = e^*|\psi_\infty|^2 \left[1 - \left(\frac{m^*\xi_{GL}(T)v_s}{\hbar}\right)^2\right] v_s$
As the superfluid velocity increases there is a deviation from linearity (i.e. $J_s \propto v_s$), with a cubic correction. If an ac superfluid velocity is induced, the current density will show a third harmonic response.

The current density as a function of $v_s$ goes through maximum at $v_s = v_c$ with $v_c^2 = \frac{\hbar^2}{3m^{*2}\xi_{GL}^2}$ and the peak is called the (de-pairing) critical current density $J_c$, $J_c = (\frac{2}{3})^{3/2}\frac{H_c}{\lambda_{eff}}$.
The result $J_c \approx \frac{H_c}{\lambda_{eff}}$ is called Silsbee's rule. It says that when the de-pairing critical current flows in a superconducting cylindrical wire, the surface field reaches the thermodynamic critical field.
The value of the order parameter at the critical current density is $|\psi_c|^2 = \frac{2}{3}|\psi_\infty|^2$.

The critical current temperature dependence near $T_c$ is $H_c(t) \propto (1-t)^{3/2}$.

### C. Appearance of Resistance in a 1D Superconducting Wire

How does resistance develop in a filamentary superconducting wire as the transition temperature is approached from below? The current flow corresponds to a twist in the phase of the order parameter with position. The GL order parameter can be written as $\psi(x) = |\psi|e^{iqx}$ which can be thought of as a helix of pitch $2\pi/q$. In the absence of an external vector potential, the superfluid velocity is simply $v_s \propto \hbar q$. The super-current is $J_s = e^*|\psi|^2 v_s \propto |\psi|^2\hbar q$.

In the zero-resistance current-carrying state a segment of the wire has constant phase $\phi_1$ at one end and a constant phase $\phi_2$ at the other, with a continuously varying-in-$x$ phase profile in between. When a voltage appears the phase difference between the two ends must increase with time at a steady rate in a manner given by the AC Josephson equation,
$\frac{d(\phi_2-\phi_1)}{dt} = \frac{2eV}{\hbar}$, where $V$ is the longitudinal voltage drop $V = V_2 - V_1$.
How does this situation develop? The voltage requires that phase is continuously "cranked in" to the phase helix. This means that an electric field will appear in the superconducting wire. The electric field will accelerate the superfluid, as we saw in the London equation, $\vec{E} = \partial(\Lambda\vec{J}_s)/\partial t$, or in other words $\frac{dv_s}{dt} = eE/m$. But we have seen that there is a limit to the acceleration, namely the critical velocity $v_c$. When this velocity is reached the solution breaks down and something new must happen. Somehow the phase windings must get "lost" to keep $v_s < v_c$ in steady state.

The new effect is created by a thermal fluctuation. Near $T_c$ there is thermal energy $k_BT$ available to be borrowed by the superconducting wire. This energy is used to suppress the magnitude of the GL order parameter in a localized region of size $\xi_{GL}$. In this case the super-current density $J_s \propto |\psi|^2\frac{d\phi}{dx} = const \propto I$ has $|\psi|^2 \to 0$ so that $\frac{d\phi}{dx}$ grows suddenly and a full $2\pi$ phase winding is lost.

This "phase slip center" acts as a local and fleeting spot to remove a $2\pi$ phase winding and establish a steady state voltage drop. The number of such phase slip events per unit time will determine the voltage.

Langer-Ambegaokar-McCumber-Halperin (LAMH) estimated that the free energy required to make such a fluctuation is $\Delta F_0 = \frac{8\sqrt{2}}{3}\mu_0 H_c^2 A\xi$, where $A$ is the cross-sectional area of the wire. This is basically the condensation energy density times the volume of the fluctuation along the wire. The resistance is found to be $\lim_{I\to 0} \frac{V}{I} = \frac{\pi\hbar^2\Omega}{2e^2 k_B T} e^{-\Delta F_0/k_B T}$, where $\Omega$ is an attempt frequency. Note that $\Delta F_0 \to 0$ as $T \to T_c$ as $\Delta F_0 \sim (1-t)^{3/2}$, hence the number of phase slips will increase as $T_c$ is approached from below. This theory is found to be in agreement with data on narrow Sn whiskers through about 5 decades in resistance, as seen on the class web site.